Distributed Implicit Discontinuous Galerkin MHD Solver

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The discontinuous Galerkin (DG) method is a favorable alternative to the finite volume (FV) method, which is often used in astrophysical codes dealing with MHD. DG methods offer higher order accuracy and reduced diffusion compared to the finite volume method while keeping the scheme highly parallelizable. The MHD equations are nonlinear, and in order not to suffer from a very small time step due to the CFL condition for stability of time discretization, we choose implicit and unconditionally stable scheme - Crank-Nicolson. Therefore, we need to solve a nonlinear problem in each time step, which involves non-differentiable numerical fluxes (such as HLLD), so care must be taken when applying the Newton's method. We propose in this work constructing the jacobian by numerical differentiation from the residual and using damped Newton's method. Another complexity of solving MHD equations using DG is satisfying zero divergence, often achieved by techniques such as divergence cleaning, or Constrained-Transport (CT). In this work, we chose another approach - using exactly divergence-free space for representation of the magnetic field. This work is being implemented using the FE libraries deal.II and Trilinos in 3D and fully parallel/distributed manner, and once finished will be available at a public software repository.

***Index Terms*—MHD, Discontinuous Galerkin, HLLD, numerical differentiation, divergence-free finite elements**

I. Introduction

There are several phenomena in the universe that we can look at as magnetohydrodynamic in nature - planets consisting of metals, interplanetary space, stars. As for the stars, these phenomena include spots, solar flares, solar winds, space weather. To study these phenomena, it is important that we are able to model them at a reasonable scale, in a reasonable detail, but most importantly - model them in a physically correct way. This means that on the path from our physical / mathematical model to numerical solutions [1], our algorithms should not spoil the solution by introducing non-physical oscillations, be in conflict with the model (having $\operatorname{div}\mathbf{B} = 0$), add artificial diffusion, etc.

II. MHD Equations, DG Method

Ideal MHD equations in the conservative form read:

$$\frac{\partial \boldsymbol{\Psi}}{\partial t} + \nabla \cdot \mathbf{F}\left(\boldsymbol{\Psi}\right) = \mathbf{S}, \tag{1}$$

where $\boldsymbol{\Psi}$ is the *state vector*, $\mathbf{F}_i,\ i = 1, 2, 3$ are the *fluxes*, and $\mathbf{S}$ is the *source term*:

$$\boldsymbol{\Psi} = \left(\rho, \pi_1, \pi_2, \pi_3, U, B_1, B_2, B_3\right), \tag{2}$$

$$\mathbf{F}_i = \begin{pmatrix} \pi_i \\ \frac{\pi_1 \pi_i}{\rho} - B_1 B_i + \frac{1}{2}\delta_{1i}\left(p + U_m\right) \\ \frac{\pi_2 \pi_i}{\rho} - B_1 B_i + \frac{1}{2}\delta_{2i}\left(p + U_m\right) \\ \frac{\pi_3 \pi_i}{\rho} - B_1 B_i + \frac{1}{2}\delta_{3i}\left(p + U_m\right) \\ \frac{\pi_i}{\rho}\left(\frac{\gamma}{\gamma - 1} p + U_k\right) + \frac{2}{\rho}\varepsilon_{ijk}\left(\pi_k B_i - \pi_i B_k\right) B_k \\ \frac{\pi_i B_1 - \pi_1 B_i}{\rho} \\ \frac{\pi_i B_2 - \pi_2 B_i}{\rho} \\ \frac{\pi_i B_3 - \pi_3 B_i}{\rho} \end{pmatrix}, \tag{3}$$

$$\mathbf{S} = \left(0, \rho g_1, \rho g_2, \rho g_3, \boldsymbol{\pi} \cdot \boldsymbol{g}, 0, 0, 0\right). \tag{4}$$

DG formulation of the resulting space-discretized problem reads

$$\begin{aligned} \int_{\Omega_t} \frac{\partial \boldsymbol{\Psi}_h}{\partial t} \mathbf{v}_h \quad &- \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}\left(\boldsymbol{\Psi}_h\right)\left(\nabla \cdot \mathbf{v}_h\right) \\ &+ \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}\left(\boldsymbol{\Psi}_h|_{ij}, \boldsymbol{\Psi}_h|_{ji}, \boldsymbol{n}_{ij}\right) \mathbf{v}_h \\ &+ \sum_{\Gamma_i \in \Gamma_B} \int_{\Gamma_i} \mathbf{H}\left(\boldsymbol{\Psi}_h|_i, \overline{\boldsymbol{\Psi}_h|_i}, \boldsymbol{n}_i\right) \mathbf{v}_h \\ &= \int_{\Omega_t} \mathbf{S}\mathbf{v}_h, \end{aligned} \tag{5}$$

where $\mathbf{v}_h$ is a test function, Γ_I is a set of all internal interfaces in the mesh, and $\Gamma_{ij} \in \Gamma_I$ an interface between two elements - K_i and K_j. Similarly Γ_B is a set of all boundary interfaces in the mesh, and $\Gamma_i \in \Gamma_B$ an interface on the boundary that neighbors the element K_i. Meaning of $\overline{\boldsymbol{\Psi}_h|_i}$ depends on the boundary conditions.

$\mathbf{H}\left(\boldsymbol{\Psi}_h|_{ij}, \boldsymbol{\Psi}_h|_{ji}, \boldsymbol{n}_{ij}\right)$ is the *numerical flux* between states $\boldsymbol{\Psi}_h|_{ij}$ and $\boldsymbol{\Psi}_h|_{ji}$ in the direction of $\boldsymbol{n}_{ij}$.

Deriving the fully (space- and time-) discretized problem using Crank-Nicolson scheme is straightforward.

III. Solving the nonlinear problem

We would like to use the Newton's method to solve the nonlinear problem arising from discretizing (5) in time using Crank-Nicolson scheme. Damped Newton's method (with damping factor α) performs iterations

$$\begin{aligned} \mathbf{J}\left(\mathbf{x}_{k+1}^n\right)\left(\Delta \mathbf{x}_{k+1}^n\right) &= -\mathbf{R}\left(\mathbf{x}_{k+1}^n\right) \\ \mathbf{x}_{k+1}^{n+1} &= \mathbf{x}_{k+1}^n + \alpha \Delta \mathbf{x}_{k+1}^n, \end{aligned} \tag{6}$$

for $n = 0, \ldots$, where $\mathbf{R}\left(\mathbf{x}_{k+1}^{n}\right)$ is the *residual*, $\mathbf{J}\left(\mathbf{x}_{k+1}^{n}\right) = \frac{\mathrm{d}\mathbf{R}\left(\mathbf{x}_{k+1}^{n}\right)}{\mathrm{d}\mathbf{x}_{k+1}^{n}}$ the *jacobian*, $\mathbf{x}_{k+1}^{0} = \mathbf{x}_k$ and $\mathbf{x}_k$ is the DG solution vector from the k-th time step. For the residual, we have

$$\begin{aligned} \mathbf{R}\left(\mathbf{x}\right)_i &= \int_{\Omega_t} \frac{\mathbf{\Psi}_h}{\Delta t} + \sum_{K_i \in T_h} \int_{K_i} \mathbf{F}\left(\mathbf{\Psi}_h\right)\left(\nabla \cdot \mathbf{v}_{hi}\right) \\ &- \sum_{\Gamma_{ij} \in \Gamma_I} \int_{\Gamma_{ij}} \mathbf{H}\left(\mathbf{\Psi}_h|_{ij}, \mathbf{\Psi}_h|_{ji}, \boldsymbol{n}_{ij}\right) \mathbf{v}_{hi} \\ &- \sum_{\Gamma_i \in \Gamma_B} \int_{\Gamma_i} \mathbf{H}\left(\mathbf{\Psi}_h|_i, \overline{\mathbf{\Psi}_h|_i}, \boldsymbol{n}_i\right) \mathbf{v}_{hi} + \int_{\Omega_t} \mathbf{S}\mathbf{v}_{hi}, \end{aligned} \tag{7}$$

where $\mathbf{v}_{hi}$ is the i-th test function, and $\mathbf{\Psi}_h$ is the (global) function corresponding to $\mathbf{x}$.

The jacobian $\mathbf{J}\left(\mathbf{x}_{k+1}^{n}\right) = \frac{\mathrm{d}\mathbf{R}\left(\mathbf{x}_{k+1}^{n}\right)}{\mathrm{d}\mathbf{x}_{k+1}^{n}}$ is calculated numerically (numerical differentiation implementation from Trilinos package Sacado is used).

The iterations in (6) are performed until $||\mathbf{R}\left(\mathbf{x}_{k+1}^{n}\right)||$ is lower than a prescribed threshold.

IV. Further Considerations

There are several topics to be dealt with when solving MHD equations numerically using the DG method. First one is the selection of numerical flux $\mathbf{H}\left(\mathbf{\Psi}_h|_{ij}, \mathbf{\Psi}_h|_{ji}, \boldsymbol{n}_{ij}\right)$ that should be as little diffusive as possible, and should satisfy the relations $\mathbf{H}\left(\mathbf{\Psi}_h, \mathbf{\Psi}_h, \boldsymbol{n}\right) = \mathbf{F}\left(\mathbf{\Psi}_h\right) \cdot \boldsymbol{n}$ and $\mathbf{H}\left(\mathbf{\Psi}_1, \mathbf{\Psi}_2, \boldsymbol{n}\right) = \mathbf{H}\left(\mathbf{\Psi}_2, \mathbf{\Psi}_1, -\boldsymbol{n}\right)$. For the ideal MHD we chose the HLLD flux designed in [9].

Another aspect of the solution, which occurs both in continuous and discontinuous FE simulations of the MHD system are spurious (nonphysical) oscillations appearing in the solution near discontinuities or sharp fronts. Here, in DG, the problem is much smaller than in the continuous case, as the instabilities occur only in regions adjacent to the sharp fronts. Our goal is to build a software package that should give a reasonable and physically correct solution to all sorts of problems, a suitable method to handle discontinuities must not change the physics (as is the case in e.g. artificial diffusion) and must not require fine-tuning parameters for it to work. Therefore we chose a parameter-less method of post-processing nature (which does not change the equations being solved). Such a method is the *Vertex-based limiter* developed in [6], and successfully applied to DG for advection-diffusion problems in [7].

Last problem, which again is present regardless of the presence of the continuity of the sought solution, is satisfying *Gauss's law*, i.e. the relationship

$$\operatorname{div} \mathbf{B} = 0, \tag{8}$$

to tackle this, an exactly divergence free FE space [4] is a mathematically clean and fully reliable method how to satisfy the relation 8, as opposed to methods of *divergence cleaning*, or *Constrained-Transport (CT)*. In order to create a generic solver, it is a very favorable method.

V. Numerical Results

We present results from one benchmark - MHD Blast - designed in [8].

The figures below are performed with a standard set of basis functions for DG - i.e. not with the exactly divergence free basis we want to use according to [4] - that is a work-in-progress currently and it shall only stabilize and qualitatively improve the results.

The following figures show the density, magnitude of momentum, magnitude of magnetic field, and the pressure in the region $-0.5 \leq x \leq 0.5; -0.5 \leq y \leq 0.5$.

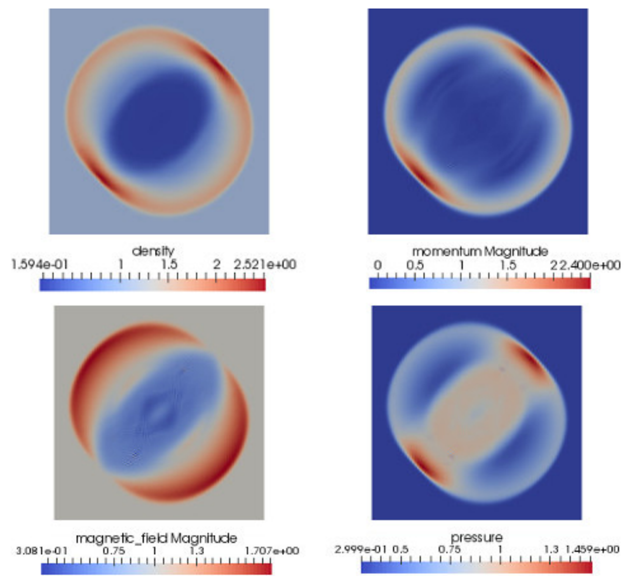

Fig. 1. MHD Blast benchmark results

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